

EXTENDING THE EXACT SEQUENCE OF NONABELIAN H^1 , USING NONABELIAN H^2 WITH COEFFICIENTS IN CROSSED MODULES

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ABSTRACT. In this note, following Dedecker and Debremaeker, we extend the cohomology exact sequence for nonabelian H^1 , using nonabelian H^2 with coefficients in crossed modules.

Let Γ be a fixed group. In this note (not to be published) we consider groups with Γ -action and crossed modules with Γ -action. Following Dedecker and Debremaeker, we extend the cohomology exact sequence for H^1 , using H^2 with coefficients in crossed modules. The obtained exact sequence seems to be essentially equivalent to the sequences of Springer [Spr66, Props. 1.27, 1.28, 1.29] and Giraud [Gir71, IV.4.2, Prop. 4.2.8], but looks nicer (more functorial) from our point of view. According to Debremaeker [Deb76], all our results are valid in the more general context of groups and crossed modules in a topos. We claim no originality.

1. SECOND COHOMOLOGY WITH COEFFICIENTS IN A CROSSED MODULE

Let $(A \xrightarrow{\rho} G)$ be a left crossed module with a Γ -action (see below). The second nonabelian cohomology with coefficients in a crossed module was considered in [Ded64], [Deb76], [Deb77f], [Br90], [Bor98], [Noo11]. We define $H^2(A \rightarrow G)$ in terms of cocycles (note that in [Bor98] this set was denoted by $H^1(\Gamma, A \rightarrow G)$, while in [Br90] the corresponding set in a more general setting was denoted by $H^1(A \rightarrow G)$). It is important that the set $H^2(A \rightarrow G)$ has a distinguished element (the *unit element*) and a distinguished subset of *neutral elements*.

Definition 1. A (left) crossed module is a homomorphism of groups $\rho: A \rightarrow G$ together with a left action $G \times A \rightarrow A$ of G on A , denoted $(g, a) \mapsto {}^g a$, such that

$$\begin{aligned} aa'a^{-1} &= \rho(a) a', \\ \rho({}^g a) &= g \cdot \rho(a) \cdot g^{-1} \end{aligned}$$

for all $a, a' \in A$, $g \in G$.

For examples of crossed modules see e.g. [Bor98, Examples 3.2.2]. Note that for any group A we have crossed modules $A \rightarrow \text{Aut } A$ and $A \rightarrow \text{Inn } A$.

2010 *Mathematics Subject Classification.* 18G50, 20J06.

Key words and phrases. Nonabelian cohomology, crossed module, exact sequence.

Partially supported by the Hermann Minkowski Center for Geometry and by the Israel Science Foundation (grant No. 870/16).

We say that our fixed group Γ acts on a crossed module $(A \rightarrow G)$ if Γ acts on A and G so that

$$\rho(\sigma a) = \sigma(\rho(a)), \quad \sigma({}^g a) = {}^g(\sigma a)$$

for all $a \in A$, $g \in G$, $\sigma \in \Gamma$.

Let $Z^2(\Gamma, A \rightarrow G)$ denote the set of pairs (u, ψ) , where

$$u: \Gamma \times \Gamma \rightarrow A \quad \text{and} \quad \psi: \Gamma \rightarrow G$$

are maps satisfying the cocycle conditions of [Bor98, (3.3.2.1-2)]:

$$\begin{aligned} u_{\sigma, \tau v} \cdot \psi_{\sigma}({}^{\sigma} u_{\tau, v}) &= u_{\sigma \tau, v} \cdot u_{\sigma, \tau} \\ \psi_{\sigma \tau} &= \rho(u_{\sigma, \tau}) \cdot \psi_{\sigma} \cdot {}^{\sigma} \psi_{\tau} \end{aligned}$$

for all $\sigma, \tau, v \in \Gamma$.

Construction 2. We define a left action of the group $\text{Maps}(\Gamma, A)$ on $Z^2(\Gamma, A \rightarrow G)$ as follows. If

$$w \in \text{Maps}(\Gamma, A), \quad (u, \psi) \in Z^2(\Gamma, A \rightarrow G),$$

then we set

$$w * (u, \psi) = (u', \psi'),$$

where

$$\begin{aligned} u'_{\sigma, \tau} &= w_{\sigma \tau} \cdot u_{\sigma, \tau} \cdot \psi_{\sigma}({}^{\sigma} w_{\tau})^{-1} \cdot w_{\sigma}^{-1}, \\ \psi'_{\sigma} &= \rho(w_{\sigma}) \cdot \psi_{\sigma} \end{aligned}$$

for all $\sigma, \tau \in \Gamma$. One checks that $(u', \psi') \in Z^2(\Gamma, A \rightarrow G)$.

We define a left action of G on $Z^2(\Gamma, A \rightarrow G)$ as follows. If

$$g \in G, \quad (u, \psi) \in Z^2(\Gamma, A \rightarrow G),$$

then we set

$$g \star (u, \psi) = (u'', \psi''),$$

where

$$\begin{aligned} u''_{\sigma, \tau} &= {}^g u_{\sigma, \tau} \\ \psi''_{\sigma} &= g \cdot \psi_{\sigma} \cdot {}^g g^{-1} \end{aligned}$$

for all $\sigma, \tau \in G$. One checks that $(u'', \psi'') \in Z^2(\Gamma, A \rightarrow G)$.

The group G acts on the left on the group $\text{Maps}(\Gamma, A)$ by

$$g \star w = w', \quad \text{where } w'_{\sigma} = {}^g w_{\sigma}$$

for $g \in G$, $w \in \text{Maps}(\Gamma, A)$, and $\sigma \in \Gamma$. We consider the semi-direct product

$$C^1(\Gamma, A \rightarrow G) := \text{Maps}(\Gamma, A) \rtimes G.$$

Then the group $C^1(\Gamma, A \rightarrow G)$ acts on the left on the set $Z^2(\Gamma, A \rightarrow G)$. Following Dedeker [Ded64] and [Ded69], we define the *thick* cohomology set and the *thin* cohomology set.

Definition 3. The *thick* cohomology set is

$$\mathbf{H}^2(A \rightarrow G) := Z^2(\Gamma, A \rightarrow G) / \text{Maps}(\Gamma, A).$$

Definition 4. The *thin* cohomology set is

$$H^2(A \rightarrow G) := Z^2(\Gamma, A \rightarrow G)/C^1(\Gamma, A \rightarrow G) = \mathbf{H}^2(A \rightarrow G)/G.$$

We have a canonical surjective map

$$(1) \quad \varkappa: \mathbf{H}^2(A \rightarrow G) \rightarrow H^2(A \rightarrow G).$$

Definition 5. The *unit cocycle* in $Z^2(\Gamma, A \rightarrow G)$ is the cocycle $(1_A, 1_G)$. The *unit classes* in $\mathbf{H}^2(A \rightarrow G)$ and $H^2(A \rightarrow G)$ are the images of the unit cocycle.

Definition 6. A *neutral cocycle* in $Z^2(\Gamma, A \rightarrow G)$ is a cocycle of the form $(1_A, \psi)$. The *neutral classes* in $\mathbf{H}^2(A \rightarrow G)$ and $H^2(A \rightarrow G)$ are the images of the neutral cocycles.

Thus the set $H^2(A \rightarrow G)$ contains the distinguished subset $H^2(A \rightarrow G)'$ of neutral elements. This subset $H^2(A \rightarrow G)'$ contains the unit element 1.

2. SECOND COHOMOLOGY WITH COEFFICIENTS IN A GROUP

Let A be a Γ -group. The Γ -action defines a homomorphism

$$f_A: \Gamma \rightarrow \text{Aut } A, \quad (f_A)_\sigma(a) = {}^\sigma a,$$

and thus it defines a Γ -kernel (Γ -band, Γ -lien)

$$\kappa_A: \Gamma \xrightarrow{f_A} \text{Aut } A \rightarrow \text{Out } A,$$

where $\text{Out } A := \text{Aut } A / \text{Inn } A$. We write $H^2(A)$ for $H^2(\Gamma, A, \kappa_A)$. The second nonabelian cohomology set $H^2(A)$ was defined by Springer [Spr66] and Giraud [Gir71], see also [Bor93], [FSS98], [Flo04], and [LA15]. By definition [Bor93, Section 1.5], the set of 2-cocycles $Z^2(\Gamma, A, \kappa_A)$ is the set of pairs (u, f) , where $u \in \text{Maps}(\Gamma \times \Gamma) \rightarrow A$ and $f \in \text{Maps}(\Gamma, \text{Aut } A)$, satisfying the 2-cocycle conditions

$$\begin{aligned} f_{\sigma\tau} &= \text{inn}(u_{\sigma,\tau}) \circ f_\sigma \circ f_\tau \\ u_{\sigma,\tau v} \cdot f_\sigma(u_{\tau,v}) &= u_{\sigma\tau,v} \cdot u_{\sigma,\tau} \\ f_\sigma &= \psi_\sigma \circ (f_A)_\sigma \quad \text{for some } \psi_\sigma \in \text{Inn } A \end{aligned}$$

for all $\sigma, \tau, v \in \Gamma$. The group $\text{Maps}(\Gamma, A)$ acts on the left on $Z^2(\Gamma, A, \kappa_A)$ as follows. If

$$w \in \text{Maps}(\Gamma, A), \quad (u, f) \in Z^2(\Gamma, A, \kappa_A),$$

then

$$w * (u, f) = (u', f'),$$

where

$$\begin{aligned} u'_{\sigma,\tau} &= w_{\sigma\tau} \cdot u_{\sigma,\tau} \cdot f_\sigma(w_\tau)^{-1} \cdot w_\sigma^{-1}, \\ f'_\sigma &= \text{inn}(w_\sigma) \circ f_\sigma \end{aligned}$$

for all $\sigma, \tau \in \Gamma$.

Definition 7. $H^2(A) = Z^2(\Gamma, A, \kappa_A) / \text{Maps}(\Gamma, A)$.

By a *neutral* cocycle in $Z^2(\Gamma, A, \kappa_A)$ we mean a cocycle of the form $(1_A, f)$, and by the *unit* cocycle we mean $(1_A, f_A)$. A *neutral class* in $H^2(A)$ is the class of a neutral cocycle, and the *unit class* 1 in $H^2(A)$ is the class of the unit cocycle. We obtain a distinguished subset $H^2(A)' \subset H^2(A)$ consisting of the neutral elements, and $H^2(A)'$ contains the unit element 1.

Note that a 2-cocycle $(u, f) \in Z^2(\Gamma, A, \kappa_A)$ defines a map $\psi: \Gamma \rightarrow \text{Inn } A$ by $\psi_\sigma = f_\sigma \circ (f_A)_\sigma^{-1}$. One checks immediately that $(u, \psi) \in Z^2(\Gamma, \text{Inn } A)$ and that the bijection

$$Z^2(\Gamma, A, \kappa_A) \xrightarrow{\sim} Z^2(\Gamma, A \rightarrow \text{Inn } A), \quad (u, f) \mapsto (u, \psi),$$

induces a canonical bijection $H^2(A) \xrightarrow{\sim} \mathbf{H}^2(A \rightarrow \text{Inn } A)$. This bijection induces a bijection $H^2(A)' \xrightarrow{\sim} \mathbf{H}^2(A \rightarrow \text{Inn } A)'$ on the set of neutral elements and takes the unit element of $H^2(A)$ to the unit element of $\mathbf{H}^2(A \rightarrow \text{Inn } A)$. We obtain a canonical surjective map

$$(2) \quad \lambda_A: H^2(A) \xrightarrow{\sim} \mathbf{H}^2(A \rightarrow \text{Inn } A) \xrightarrow{\simeq} H^2(A \rightarrow \text{Inn } A).$$

Theorem 8 (Debremaeker [Deb76, Ch. V, Thm. 3, p. 112]). *The canonical surjection (2) is a bijection.*

First proof. Let Z_A denote the center of A , which is a Γ -group. Then the group of 2-cocycles $Z^2(\Gamma, Z_A)$ acts on the left on the set $Z^2(\Gamma, A, \kappa_A)$ as follows:

$$\text{if } z \in Z^2(\Gamma, Z_A), (u, f) \in Z^2(\Gamma, A, \kappa_A), \text{ then } z * (u, f) = (zu, f).$$

This action induces an action of $H^2(Z)$ on $H^2(A)$, which is simply transitive, see [ML63, IV-Thm. 8.8] or [Spr66, Prop. 1.17].

On the other hand, the group $Z^2(\Gamma, Z_A)$ acts on the left on the set $Z^2(\Gamma, A \rightarrow \text{Inn } A)$ as follows:

$$\text{if } z \in Z^2(\Gamma, Z_A), (u, \psi) \in Z^2(\Gamma, A \rightarrow \text{Inn } A), \text{ then } z * (u, \psi) = (zu, \psi).$$

This map induces an action of $H^2(Z_A)$ on $H^2(A \rightarrow \text{Inn } A)$ and, by the action of $H^2(Z_A)$ on the unit element $1 \in H^2(A \rightarrow \text{Inn } A)$, it induces a map

$$\mu: H^2(Z_A) \rightarrow H^2(A \rightarrow \text{Inn } A),$$

which can be factored as

$$\mu: H^2(Z_A) \xrightarrow{\sim} H^2(Z_A \rightarrow 1) \xrightarrow{\iota_*} H^2(A \rightarrow \text{Inn } A),$$

where the map ι_* is induced by the embedding of crossed modules

$$\iota: (Z \rightarrow 1) \hookrightarrow (A \rightarrow \text{Inn } A).$$

Since the embedding ι is a quasi-isomorphism of crossed modules, the map ι_* is bijective (see [Bor92, Thm. 3.3]), hence the map μ is bijective and the action of $H^2(Z_A)$ on $H^2(A \rightarrow \text{Inn } A)$ is simply transitive.

Now, since the map (2) is $H^2(Z_A)$ -equivariant, we conclude that it is bijective, as required.

Second proof (similar to [Spr66, Proof of Prop. 1.19]). We wish to prove that the surjective map

$$\simeq: \mathbf{H}^2(A \rightarrow \text{Inn } A) \rightarrow H^2(A \rightarrow \text{Inn } A)$$

is bijective. It suffices to show that $\text{Inn } A$ acts on $\mathbf{H}^2(A \rightarrow \text{Inn } A)$ trivially.

Let

$$g \in \text{Inn } A, \quad g = \text{inn}(b), \quad b \in A.$$

One can check that the cocycles $g \star (u, \psi)$ and (u, ψ) give the same class in $\mathbf{H}^2(A \rightarrow \text{Inn } A)$, namely, that

$$g \star (u, \psi) = w * (u, \psi),$$

where

$$w \in \text{Maps}(\Gamma, A), \quad w_\sigma = b \cdot \psi_\sigma({}^\sigma b)^{-1} \quad \text{for } \sigma \in \Gamma.$$

This completes the second proof. \square

3. COHOMOLOGY EXACT SEQUENCE

Let

$$(3) \quad 1 \rightarrow A \xrightarrow{i} B \xrightarrow{j} C \rightarrow 1$$

be a short exact sequence of Γ -groups. We construct a connecting map

$$\Delta: H^1(C) \rightarrow H^2(A \rightarrow \text{Inn } B) \quad (\text{sic!}).$$

Let $c \in Z^1(\Gamma, C) \subset \text{Maps}(\Gamma, C)$. We lift c to some map $b: \Gamma \rightarrow B$ and define

$$\begin{aligned} u_{\sigma, \tau} &= b_{\sigma\tau} \cdot {}^\sigma b_\tau^{-1} \cdot b_\sigma^{-1} \in A \\ \psi_\sigma &= \text{inn}(b_\sigma) \in \text{Inn } B \end{aligned}$$

for $\sigma, \tau \in \Gamma$. We set $\Delta([c]) = [u, \psi]$, where $[c]$ denotes the class in $H^1(C)$ of c , and $[u, \psi]$ denotes the class in $H^2(A \rightarrow \text{Inn } B)$ of $(u, \psi) \in Z^2(\Gamma, A \rightarrow \text{Inn } B)$. One checks that $(u, \psi) \in Z^2(\Gamma, A \rightarrow \text{Inn } B)$ and that the map Δ is well defined.

Consider the morphisms of crossed modules

$$(A \rightarrow \text{Inn } B) \xrightarrow{i_*} (B \rightarrow \text{Inn } B) \xrightarrow{j_*} (C \rightarrow \text{Inn } C)$$

and the sequence

$$(4) \quad H^1(B) \xrightarrow{j_*} H^1(C) \xrightarrow{\Delta} H^2(A \rightarrow \text{Inn } B) \xrightarrow{i_*} H^2(B \rightarrow \text{Inn } B) \xrightarrow{j_*} H^2(C \rightarrow \text{Inn } C)$$

Theorem 9 (Dedecker [Ded69, Thm. 2.2] and Debremaeker [Deb76, Ch. IV, Thm. 2.1.7, p. 103]). *For an exact sequence of Γ -groups (3), the sequence (4) is exact in the following sense:*

- (i) *an element of $H^1(C)$ is contained in the image of $H^1(B)$ if and only if its image in $H^2(A \rightarrow \text{Inn } B)$ is neutral;*
- (ii) *an element of $H^2(A \rightarrow \text{Inn } B)$ is contained in the image of $H^1(C)$ if and only if its image in $H^2(B \rightarrow \text{Inn } B)$ is the unit element;*
- (iii) *an element of $H^2(B \rightarrow \text{Inn } B)$ is contained in the image of $H^2(A \rightarrow \text{Inn } B)$ if and only if its image in $H^2(C \rightarrow \text{Inn } C)$ is neutral.*

Proof. Let $b \in Z^1(\Gamma, B)$. We show that $\Delta \circ j_*$ takes $[b]$ to a neutral class. Indeed, by the definition of Δ , the composite map $\Delta \circ j_*$ maps $[b]$ to $[u^A, \psi^A]$ where

$$u_{\sigma, \tau}^A = b_{\sigma\tau} \cdot {}^\sigma b_\tau^{-1} \cdot b_\sigma^{-1} = 1$$

because b is a cocycle. Thus $[u^A, \psi^A] = [1, \psi^A]$ is a neutral class.

Conversely, let $c \in Z^1(\Gamma, C)$ and assume that Δ takes $[c]$ to a neutral class. Let us lift c to some map $b: \Gamma \rightarrow B$. Then $\Delta[c] = [u^A, \psi^A]$, where

$$u_{\sigma, \tau}^A = b_{\sigma\tau} \cdot {}^\sigma b_\tau^{-1} \cdot b_\sigma^{-1},$$

$$\psi_\sigma^A = \text{inn}(b_\sigma).$$

By assumption $[u^A, \psi^A]$ is a neutral class in $H^2(A \rightarrow \text{Inn } B)$, i.e., there exists a map $a: \Gamma \rightarrow A$ such that

$$a * (u^A, \psi^A) = (1, \psi'^A).$$

This means that

$$a_{\sigma\tau} \cdot b_{\sigma\tau} \cdot {}^\sigma b_\tau^{-1} \cdot b_\sigma^{-1} \cdot b_\sigma \cdot {}^\sigma a_\tau^{-1} \cdot b_\sigma^{-1} \cdot a_\sigma^{-1} = 1.$$

that is,

$$a_{\sigma\tau} b_{\sigma\tau} \cdot {}^\sigma (a_\tau b_\tau)^{-1} \cdot (a_\sigma b_\sigma)^{-1} = 1.$$

Set $b'_\sigma = a_\sigma b_\sigma$, then $b'_{\sigma\tau} = b'_\sigma \cdot {}^\sigma b'_\tau$, hence b' is a cocycle. Clearly j_* takes $[b']$ to $[c]$, and therefore, $[c] \in \text{im } j_*$, as required.

Let $c \in Z^1(\Gamma, C)$. We show that $i_* \circ \Delta$ takes $[c]$ to 1. Indeed, let us lift c to some map $b: \Gamma \rightarrow B$. Then the composite map $i_* \circ \Delta$ takes $[c]$ to the class $[u^B, \psi^B]$ where

$$u_{\sigma, \tau}^B = b_{\sigma\tau} \cdot {}^\sigma b_\tau^{-1} \cdot b_\sigma^{-1} \in B$$

$$\psi_\sigma^B = \text{inn}(b_\sigma) \in \text{Inn } B,$$

and clearly $(u^B, \psi^B) = b * (1, 1)$, hence $[u^B, \psi^B] = [1, 1]$.

Conversely, let $[u^A, \psi^A] \in H^2(A \rightarrow \text{Inn } B)$ and assume that $i_*([u^A, \psi^A]) = [1, 1]$. Clearly $i_*([u^A, \psi^A]) = [u^A, \psi^A]$, so we obtain that

$$[u^A, \psi^A] = [1, 1] \in H^2(B \rightarrow \text{Inn } B).$$

By Theorem 8 we have $H^2(B \rightarrow \text{Inn } B) = \mathbf{H}^2(B \rightarrow \text{Inn } B)$, hence $(u^A, \psi^A) = b * (1, 1)$ for some $b: \Gamma \rightarrow B$. We have

$$u_{\sigma, \tau}^A = b_{\sigma\tau} \cdot {}^\sigma b_\tau^{-1} \cdot b_\sigma^{-1},$$

$$\psi_\sigma^A = \text{inn}(b_\sigma).$$

Set $c = j \circ b: \Gamma \rightarrow C$. Since $u_{\sigma, \tau}^A \in A$, we see that

$$c_{\sigma\tau} \cdot {}^\sigma c_\tau^{-1} \cdot c_\sigma^{-1} = 1,$$

hence c is a cocycle. Clearly, $[u^A, \psi^A] = \Delta([c])$. Thus $[u^A, \psi^A] \in \text{im } \Delta$, as required.

Let $(u^A, \psi^A) \in Z^2(\Gamma, A \rightarrow \text{Inn } B)$. We show that $j_* \circ i_*$ takes $[u^A, \psi^A]$ to a neutral class. Indeed, for any $\sigma, \tau \in \Gamma$ we have $u_{\sigma, \tau}^A \in A$. It follows that the image of $[u^A, \psi^A]$ under the composite map $j_* \circ i_*$ is of the form $[1, \psi^C]$, and hence is neutral.

Conversely, assume that j_* takes $[u^B, \psi^B]$ to a neutral class $[u^C, \psi^C]$. This means that there exists a map $c: \Gamma \rightarrow C$ such that $c * (u^C, \psi^C) = (1, \psi^C)$. Let us lift c to a map $b: \Gamma \rightarrow B$ and set $(u'^B, \psi'^B) = b * (u^B, \psi^B)$. Then for any $\sigma, \tau \in G$ we have $u'_{\sigma, \tau} \in A$. We see that $[u^B, \psi^B] = [u'^B, \psi'^B]$ lies in the image of i_* , as required. This completes the proof of the theorem. \square

4. A VERSION OF THEOREM 9.

We write

$$G = (\text{Inn } B)|_A := \{\text{inn}(b)|_A \mid b \in B\},$$

the group of restrictions to A of the inner automorphisms of B . Then $G \subset \text{Aut } A$. We have an epimorphism $\text{Inn } B \rightarrow G$ and a morphism of crossed modules

$$\pi: (A \rightarrow \text{Inn } B) \rightarrow (A \rightarrow G).$$

Lemma 10. *For $(u, \psi) \in Z^2(\Gamma, A \rightarrow \text{Inn } B)$, its class $[u, \psi] \in H^2(A \rightarrow \text{Inn } B)$ is neutral if and only if $\pi_*([u, \psi]) \in H^2(A \rightarrow G)$ is neutral.*

Proof. Easy. \square

Corollary 11. *For an exact sequence of Γ -groups (3), the sequence*

$$(5) \quad H^1(B) \xrightarrow{j_*} H^1(C) \xrightarrow{\Delta \circ \pi_*} H^2(A \rightarrow G)$$

is exact in the following sense: a cohomology class $c \in H^1(C)$ comes from $H^1(B)$ if and only if its image in $H^2(A \rightarrow G)$ is neutral.

5. EXAMPLE

We compute the map $\Delta \circ \pi_*: H^1(C) \rightarrow H^2(A \rightarrow G)$ in the case when A is abelian. Here $G = (\text{Inn } B)|_A$.

Let $(u, \psi) \in Z^2(\Gamma, A \rightarrow G)$, then

$$\begin{aligned} u_{\sigma, \tau v} \cdot \psi_{\sigma}(u_{\tau, v}) &= u_{\sigma \tau, v} \cdot u_{\sigma, \tau} \\ \psi_{\sigma \tau} &= \text{inn}(u_{\sigma, \tau}) \cdot \psi_{\sigma} \cdot {}^{\sigma}\psi_{\tau}. \end{aligned}$$

Since A is abelian, the homomorphism $A \rightarrow G$ is trivial, hence ψ is a 1-cocycle, $\psi \in Z^1(\Gamma, G)$. Moreover, u is a 2-cocycle, $u \in Z^2(\Gamma, {}_{\psi}A)$. One checks immediately that the map

$$Z^2(\Gamma, A \rightarrow G) \rightarrow Z^1(\Gamma, G), \quad (u, \psi) \mapsto \psi$$

induces a surjective map

$$\zeta: H^2(A \rightarrow G) \rightarrow H^1(G), \quad [u, \psi] \mapsto [\psi].$$

Moreover, for given $\psi \in Z^1(\Gamma, G)$, we have a bijection

$$\lambda_{\psi}: H^2({}_{\psi}A) \xrightarrow{\sim} \zeta^{-1}([\psi]), \quad [u] \mapsto [u, \psi],$$

and $\lambda_{\psi}([u])$ is neutral in $H^2(A \rightarrow G)$ if and only if $[u] = 0 \in H^2({}_{\psi}A)$.

Since A is abelian, C acts on A , and we obtain a surjective homomorphism $p: C \rightarrow G$. Let $c \in Z^1(\Gamma, C)$, $\psi = p_*(c) \in Z^1(\Gamma, G)$, then we write ${}_cA$ for ${}_pA$.

Let us lift $c: \Gamma \rightarrow C$ to some map $b: \Gamma \rightarrow B$ and set

$$u_{\sigma, \tau} = b_{\sigma\tau} \cdot {}^{\sigma}b_{\tau}^{-1} \cdot b_{\sigma}^{-1},$$

then $u \in Z^2(\Gamma, {}_cA)$. Set

$$\Delta_S(c) = [u] \in H^2({}_cA).$$

A simple computation shows that the image of $[c]$ in $H^2(A \rightarrow G)$ is

$$\lambda_{\psi}(\Delta_S(c)) \in \zeta^{-1}([\psi]) \subset H^2(A \rightarrow G),$$

where $\psi = p_*(c) \in Z^1(\Gamma, G)$, $\Delta_S(c) \in H^2({}_cA)$. This image is a neutral class in $H^2(A \rightarrow G)$ if and only if $\Delta_S(c) = 0$.

Applying Corollary 11, we recover a result of Serre [Se94, I.5.6, Prop. 41]: a cohomology class $[c] \in H^1(C)$ comes from $H^1(B)$ if and only if

$$\Delta_S(c) = 0 \in H^2({}_cA).$$

Remark 12. This note was inspired by the paper [Dun16] of Alexander Duncan, who in Section 7 constructed "by hand" the cohomology set $H^2(A \rightarrow G)$ with a distinguished subset of neutral elements, where A is an *abelian* group, G is a subgroup of $\text{Aut } A$, and the homomorphism $A \rightarrow G$ is trivial.

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